An expression for the rise time of a logistic function is derived, where rise time is defined to be the time taken for a logistic to rise from 10% to 90% of the final value.

An expression for the average slope between the 10% and 90% of final value marks is also derived.

The Logistic Equation can be written

\[ P(t) = \frac{\beta}{\delta + (\beta - \delta) \cdot e^{-\beta \cdot (t-t_0)}} \]  ... Eqn (1)

where

- \( t_0 \) = time of first case
- \( P(t_0) = 1 \) and
- \( P(t = \infty) = \frac{\beta}{\delta} \) since second (exponential) term on denominator \( \rightarrow 0 \)

define \( \text{rise time} = \) time for \( P(t) \) to rise from 10% to 90% of \( P(\infty) \)

Let

\( t_a \) = time at which \( P(t) = a \cdot P(\infty) \)

\( Q(t) = (\beta - \delta) \cdot e^{-\beta \cdot (t-t_0)} \)

then at \( t_a \),

\[ P(t_a) = \frac{\beta}{\delta + Q(t_a)} = a \cdot \frac{\beta}{\delta} \]

solving...

\[ Q(t_a) = \frac{\delta \cdot (1-a)}{a} \]
Rise Time of Logistic Equation

\[ (\beta - \delta) \cdot e^{-\beta \cdot (t_a - t_0)} = \frac{\delta \cdot (1 - a)}{a} \]

Solving for \( t_a \):

\[ e^{-\beta \cdot (t_a - t_0)} = \frac{\delta \cdot (1 - a)}{(\beta - \delta) \cdot a} \]

\[ -\beta \cdot (t_a - t_0) = \ln \left[ \frac{\delta (1 - a)}{(\beta - \delta) \cdot a} \right] \]

\[ t_a = t_0 + \frac{1}{\beta} \ln \left[ \frac{1 - a}{(1 - \beta/\delta) \cdot a} \right] \]  \[ \text{Eqn 3} \]

\[ \therefore \text{Rise Time} = t_{0.9} - t_{0.1} = \frac{1}{\beta} \left\{ \ln \left[ \frac{1}{9(\beta/\delta - 1)} \right] - \ln \left[ \frac{9}{(\beta/\delta - 1)} \right] \right\} \]

but

\[ \ln (\frac{a}{b}) = \ln (a) - \ln (b) \]

so

\[ |\text{Rise Time}| = \frac{1}{\beta} \ln \left( \frac{1}{81} \right) = \frac{\ln (81)}{\beta} \]

It can also be shown that the slope between 10% and 90% levels is:

\[ \text{Rise Slope} = \frac{0.8 \cdot \frac{\beta}{\delta}}{\ln (81)} = \frac{0.8 \cdot \beta^2}{\ln (81) \cdot \delta} = 0.182 \cdot \frac{\beta^2}{\delta} \]  \[ \text{(cases per day)} \]

For studying epidemics, infection spread occurs over several days. A curve which rose over just one day might be sampling a single source or too small a number of sources to be reliable as an indication of how the epidemic will progress.

To eliminate this possibility, the search for a fitting epidemic curve should be constrained.

A constraint like

\[ \beta < \ln (81) \]

will ensure this (\( \ln (81) = 2.48 \)).

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