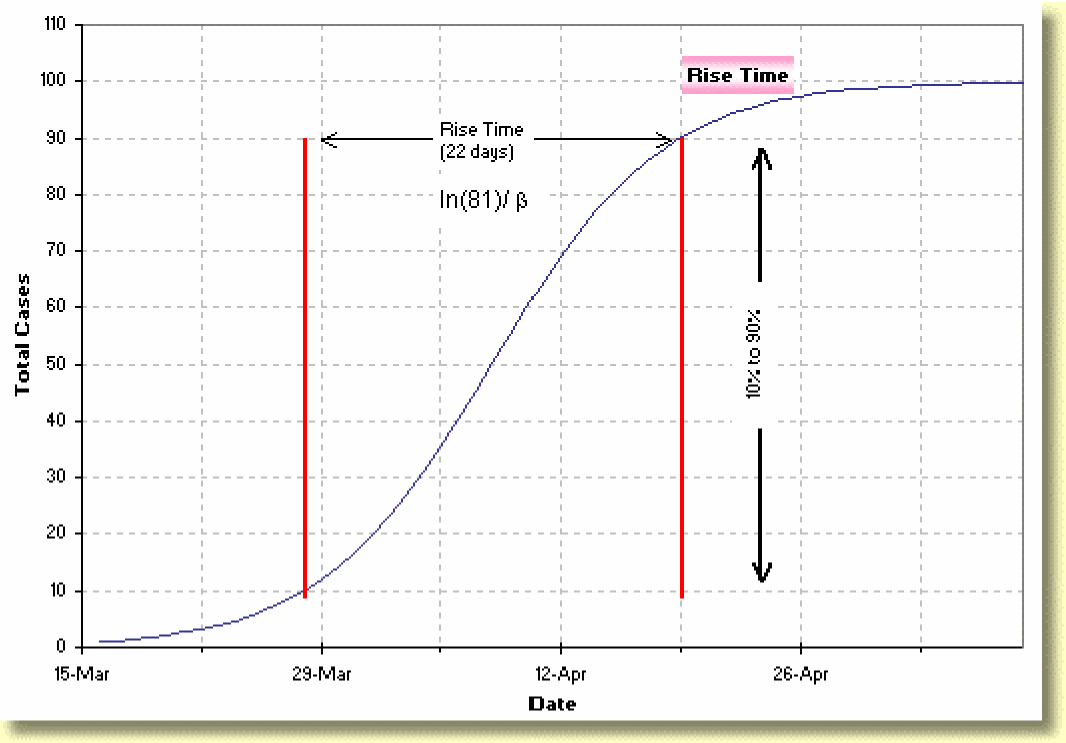
Rise Time of Logistic Equation

An expression for the rise time of a logistic function is derived, where rise time is defined to be the time taken for a logistic to rise from 10% to 90% of the final value.

An expression for the average slope between the 10% and 90% of final value marks is also derived.



The Logistic Equation can be written

$$P(t) = \frac{\beta}{\delta + (\beta - \delta)\cdot e^{-\beta\cdot(t-t_0)}} \qquad \text{... Eqn (1)}$$

where

t_0 = time of first case

$P(t_0) = 1$ and

$P(t = \infty) = \dfrac{\beta}{\delta}$ since second (exponential) term on denominator $\rightarrow 0$)

define *rise time* = time for P(t) to rise from 10% to 90% of $P(\infty)$

Let

t_a = time at which $P(t) = a \cdot P(\infty)$

$Q(t) = (\beta - \delta)\cdot e^{-\beta\cdot(t-t_0)}$

then at t_a,

$$P(t_a) = \frac{\beta}{\delta + Q(t_a)} = a\cdot\frac{\beta}{\delta}$$

solving...

$$\therefore \quad Q(t_a) = \frac{\delta\cdot(1-a)}{a}$$

$$\therefore (\beta-\delta)\cdot e^{-\beta\cdot(t_a-t_0)} = \frac{\delta\cdot(1-a)}{a}$$

Solving for t_a :

$$\therefore e^{-\beta\cdot(t_a-t_0)} = \frac{\delta\cdot(1-a)}{(\beta-\delta)\cdot a}$$

$$\therefore -\beta\cdot(t_a-t_0) = \ln\left[\frac{\delta(1-a)}{(\beta-\delta)\cdot a}\right] \qquad \text{(ln is logarithm base e)}$$

$$\therefore t_a = t_0 + \frac{1}{\beta}\cdot\ln\left[\frac{(1-a)}{(\beta/\delta-1)\cdot a}\right] \qquad \text{... Eqn 3}$$

$$\therefore \text{Rise Time} = t_{0.9} - t_{0.1} = \frac{1}{\beta}\cdot\left\{\ln\left[\frac{1}{9(\beta/\delta-1)}\right]-\ln\left[\frac{9}{(\beta/\delta-1)}\right]\right\}$$

but

$$\ln(a/b) = \ln(a)-\ln(b),$$

so

$$|\text{Rise Time}| = \left|\frac{1}{\beta}\ln(\frac{1}{81})\right| = \left|\frac{\ln(81)}{\beta}\right|$$

It can also be shown that the slope between 10% and 90% levels is :

$$\text{Rise Slope} = \frac{0.8\cdot\beta/\delta}{\frac{\ln(81)}{\beta}} = \frac{0.8}{\ln(81)}\cdot\frac{\beta^2}{\delta} = 0.182\cdot\frac{\beta^2}{\delta} \qquad \text{(cases per day)}$$

For studying epidemics, infection spread occurs over several days. A curve which rose over just one day might be sampling a single source or too small a number of sources to be reliable as an indication of how the epidemic will progress

To eliminate this possibility, the search for a fitting epidemic curve should be constrained.

A constraint like

$$\beta < \ln(81)$$

will ensure this (ln (81) = 2.48).

Author: K Duffy, May 12th, 2003